PHILOSOPHICAL TRANSACTIONS A

royalsocietypublishing.org/journal/rsta

Review



Cite this article: Hart JD, Larger L, Murphy TE, Roy R. 2019 Delayed dynamical systems: networks, chimeras and reservoir computing. *Phil. Trans. R. Soc. A* **377**: 20180123. http://dx.doi.org/10.1098/rsta.2018.0123

Accepted: 3 April 2019

One contribution of 14 to a theme issue 'Nonlinear dynamics of delay systems'.

Subject Areas: complexity, optics

Keywords:

networks, reservoir computing, chimeras, delayed dynamical system

Author for correspondence:

Rajarshi Roy e-mail: rroy@umd.edu

Delayed dynamical systems: networks, chimeras and reservoir computing

Joseph D. Hart^{1,2}, Laurent Larger⁵, Thomas E. Murphy^{1,3} and Rajarshi Roy^{1,2,4}

¹Institute for Research in Electronics and Applied Physics, ²Department of Physics, ³Department of Electrical and Computer Engineering, and ⁴Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA ⁵FEMTO-ST Institute/Optics Department, CNRS and University Bourgogne Franche-Comté, 15B avenue des Montboucons, 25030 Besançon Cedex, France

JDH, 0000-0002-9396-0574; LL, 0000-0003-1245-1602; TEM, 0000-0002-8286-3832; RR, 0000-0002-2381-829X

We present a systematic approach to reveal the correspondence between time delay dynamics and networks of coupled oscillators. After early demonstrations of the usefulness of spatio-temporal representations of time-delay system dynamics, extensive research on optoelectronic feedback loops has revealed their immense potential for realizing complex system dynamics such as chimeras in rings of coupled oscillators and applications to reservoir computing. Delayed dynamical systems have been enriched in recent years through the application of digital signal processing techniques. Very recently, we have showed that one can significantly extend the capabilities and implement networks with arbitrary topologies through the use of field programmable gate arrays. This architecture allows the design of appropriate filters and multiple time delays, and greatly extends the possibilities for exploring synchronization patterns in arbitrary network topologies. This has enabled us to explore complex dynamics on networks with nodes that can be perfectly identical, introduce parameter heterogeneities and multiple time delays, as well as change network topologies to control the formation and evolution of patterns of synchrony.

This article is part of the theme issue 'Nonlinear dynamics of delay systems'.

1. Introduction

Networks of coupled oscillators are dynamical systems of great interest for both basic and applied research. Networks are high-dimensional systems that can display a great variety of dynamical behaviours. Applications abound, from neuroscience [1] and gene regulation [2] to the power grid [3] and machine learning [4]. Networks have long been a fertile ground for theoretical research [5]; however, experiments on large networks have proven difficult because of the necessity to create, connect and measure a large number of independent oscillators. In the few cases where experiments with large networks have been possible, it is often difficult or impossible to reconfigure the network, with a few notable exceptions [6–8].

Nonlinear systems with time-delayed feedback are a different type of high-dimensional dynamical system that are much easier to study experimentally. Time delays often arise when the intrinsic dynamics of a system are fast enough that the finite propagation velocity of signals must be taken into account. For example, in a semiconductor laser with time-delayed feedback through an external mirror, the photon lifetime is significantly shorter than the feedback time, which can cause the laser intensity to oscillate chaotically [9]. From an experimental point of view, delay systems are particularly attractive because the complexity of the dynamics usually increases with the delay [10–12], which is typically easy to control.

The simplest delay systems can be modelled by [13]

$$\tau_L \dot{x}(t) = -x(t) + F(x(t - \tau_D)), \tag{1.1}$$

where τ_L is the intrinsic time scale of the system, F(x) is a nonlinear function of x and τ_D is the time delay. Equation (1.1) has been used to model systems from many different areas of science [14], including physiology [15], population dynamics [16] and laser physics [17]. Systems described by equation (1.1) have been shown to display a wide variety of interesting behaviours, including square waves [18,19], new types of chaos (in the case that τ_D varies in time) [20] and spatio-temporal phenomena [21].

Indeed, research over the last 25 years has shown that a wide variety of spatio-temporal phenomena can be observed in temporal systems with a long delayed feedback. The interpretation of dynamics in delayed systems as spatio-temporal phenomena is enabled by the space-time representation [22]. Some of the theoretically predicted and experimentally observed spatio-temporal phenomena include defect-mediated turbulence [23,24], coarsening [25,26], domain nucleation [27], spatial coherence resonance [28] and phase transitions [29].

Our focus in this paper is on the implementation of networks of truly identical coupled oscillators through the use of a single nonlinear delayed feedback system. This is made possible through the same space–time representation that led to the observation of other spatio-temporal phenomena in delay systems. Originally developed for the implementation of neural networks for reservoir computing in hardware [30,31], this technique for implementing networks has subsequently been adapted for basic research, such as the study of chimera states in ring networks [32,33] and cluster synchronization in arbitrary networks [34–36]. This framework for implementing networks is particularly attractive, because it allows for experiments on large networks without building a large number of separate physical oscillators and it allows for experiments on truly identical oscillators. We focus on opto-electronic implementations, which are popular due to their speed, cost and ease of implementation; however, the techniques described are applicable to other delay systems as well.

In §2, we introduce a basic mathematical description of a delayed feedback system through a commonly used integro-differential delay equation. Additionally, we present a less commonly used, but equivalent, description from filter theory that employs a convolution integral of the feedback signal with the impulse response that describes the bandwidth limitations of the system. This second formalism, when viewed in the space–time representation, provides insight into how networks of oscillators can be realized with a single nonlinear system with delayed feedback. Finally, we describe one particular opto-electronic oscillator that has been a favourite of experimenters due to its reliability and ease of implementation over a wide range of parameters and time scales.

The space-time representation of delay systems is presented in §3. The space-time representation relies on the separation of time scales—fast dynamics and a long delay—to parametrize time as a time-like integer number that counts the number of round-trip times and a continuous, space-like variable that denotes the position within each delay. This analogy between feedback systems with a long time delay and spatio-temporal systems has allowed for a deeper understanding of many complex phenomena observed in delay systems, including defect-mediated turbulence [23,24], coarsening [25,26], domain nucleation [27], spatial coherence resonance [28], phase transitions [29] and now, network dynamics.

Section 4 describes in detail how the space-time representation allows for the implementation of networks of truly identical coupled oscillators using only a single delayed feedback system. Traditional networks are spatially multiplexed: all nodes are updated simultaneously in parallel depending on their previous states. Delay feedback networks replace the spatial multiplexing of traditional networks with temporal multiplexing, in which the single nonlinear element serially updates the nodes, which are distributed across the delay line. The nodes are coupled together by the 'inertia', or non-instantaneous response time, of the system, which arises from the bandwidth limitations of the components. When this filtering is time-invariant, the resulting network has cyclic symmetry. In particular, §4 focuses on the discrete-time case; e.g. when the time delay is implemented by a digital delay line.

The use of delay networks for hardware implementations of reservoir computers is discussed in §5. Reservoir computing (RC)—alternatively echo state networks [4] or nonlinear transient computing [37]—is a type of neural network in which only the output connections are trained (the input and internal connections are fixed). Reservoir computers are particularly attractive because they can be trained by simple linear regression and because they are well suited for implementation in specialized hardware. Delay networks have proven to be particularly well suited for hardware implementations of RC.

Section 6 extends the delay network formalism developed in §4 to the continuous-time case (the case of analogue delay lines).

Chimera states are an unexpected coexistence of spatial domains of coherence and incoherence in a system of identical oscillators with symmetric coupling [38,39]. Chimera states were particularly difficult to observe in experiments because they typically (but not always [40,41]) occur in large networks, which are difficult to experimentally implement. Initially observed in 2012 [6,7] a decade after their prediction, they were soon after observed in electronic [32] and opto-electronic [33] delay systems, as presented in §7.

A recently developed technique [34] that allows a network with any topology to be implemented in a delay system is described in §8. This technique replaces the time-invariant filters used in the original delay network implementations with a time-dependent filter. The time-dependent filter, implemented digitally with a field-programmable gate array (FPGA), extends the range of networks that can be realized from only networks with rotational symmetry to networks with completely arbitrary topology.

Introduction to opto-electronic oscillators with delayed feedback

The basic form of a delayed feedback system is depicted by the block diagram in figure 1*a*. The output of a nonlinearity $F(\cdot)$ is amplified, filtered and delayed before being fed back as the input to the nonlinearity. The filtering may either be intentionally implemented or arise from the bandwidth limitations of the system. Such a delayed feedback system can be described by the convolution of the input to the filter with the impulse response h(t) that characterizes the filter [42]:

$$x(t) = h(t) * \beta F(x(t - \tau_D)) = \beta \int_{-\infty}^{\infty} h(t - t') F(x(t' - \tau_D)) dt' = \beta \int_{-\infty}^{t} h(t - t') F(x(t' - \tau_D)) dt', \quad (2.1)$$



Figure 1. Nonlinear delayed feedback system. (*a*) Block diagram of a delay system. $v(t) = \beta F(x(t - \tau_D))$ is the input to the linear filter described by the impulse response h(t), and x(t) is the filter output. (*b*) Experimental set-up of an opto-electronic oscillator delayed feedback system. The filtering is performed either by the component with the narrowest bandwidth (usually the photodiode) or by a stand-alone filter (not shown). The oscillator can be a discrete-time map (when powered by a pulsed laser) or a continuous-time system (when powered by a CW laser). (Online version in colour.)

where in the last step we use the property that h(t) is causal. In equation (2.1), x(t) is the filter output, β is the round trip gain, and τ_D is the time delay.

For a large class of filters, an equivalent delay differential equation can be used to describe the system [42]. In the case that the bandwidth limitations of the system can be accurately described by a two-pole bandpass filter, the delay differential equation is

$$\tau_L \dot{x}(t) = -\left(1 + \frac{\tau_L}{\tau_H}\right) x(t) - \frac{1}{\tau_H} \int_{-\infty}^t x(s) \,\mathrm{d}s + \beta F(x(t - \tau_D)), \tag{2.2}$$

where τ_D is the time delay, $\tau_L = 1/2\pi f_L$ is the low-pass filter response time, and $\tau_H = 1/2\pi f_H$ is the high-pass filter response time. Equation (2.2) is quite general in that it can be used to model many delayed feedback systems. Indeed, by considering the limit $\tau_H \rightarrow \infty$ (i.e. the case of a low-pass instead of a band-pass filter), equation (2.2) reduces to equation (1.1).

One experimental system of particular interest that can be accurately modelled by equation (2.2) is the opto-electronic oscillator. Opto-electronic oscillators were originally studied in bulk optics [43] and soon after implemented using standard telecommunications components [44]. These systems have been found to be extremely rich in their dynamics, in part because they can span an enormous range of time scales [45]. They have been used to study chaotic breathers [46], broadband chaos [47], network dynamics [41,48] and the transition from noise to chaos [49]. Additionally, opto-electronic oscillators are useful for a variety of applications, including the generation of high-spectral purity microwaves [50], chaos communications [51,52] and RC [31,37].

A schematic of an opto-electronic oscillator is shown in figure 1*b*. Constant intensity light from a fibre-coupled CW laser passes through an integrated electro-optic Mach–Zehnder intensity modulator, which provides the nonlinearity $F(x) = \sin^2(x + \phi)$. The quantity x(t) represents the normalized voltage applied to the intensity modulator, and ϕ is the normalized DC bias voltage. The time delay is implemented by an optical or electronic (not shown) delay line. The filtering is performed either by the photodiode (the component with the narrowest bandwidth) or a standalone analogue [50] or digital [53] filter (not shown). For a recent review of these opto-electronic oscillators, see [54].

Alternatively, the system can be turned into a discrete-time map by pulsing the laser at a repetition rate $f_r = N/\tau_D$ [55]. In this case, the system can be modelled as

$$x[k] = \beta \sum_{m=-\infty}^{k} h[k-m]F(x[m-N]).$$
(2.3)

where x[k] is the height of the *k*th electrical pulse applied to the modulator, *h* is the infinite impulse response of the filter sampled at the repetition rate f_r . As the repetition rate $f_r \rightarrow \infty$, time becomes continuous, the sum becomes a convolution integral, and we obtain equation (2.1). Therefore, this system allows for the study of the transition from discrete to continuous time in chaotic systems.

3. Space-time representation

The space–time representation of delay systems was originally motivated by the numerical treatment of delay differential equations [10]. The time variable is split up into a continuous variable σ bounded between 0 and τ_D , and an independent discrete variable *n* that counts the number of delays since the origin. Ikeda & Matsumoto [56] were the first to consider σ to be a 'spatial' variable in their modelling of optical turbulence. The space–time representation was formalized and first used on experimental data by Arecchi *et al.* in 1992 [22] in order to study long-time correlations on the order of one delay in a CO₂ laser with delayed feedback. Since then, the relationship between delay systems and spatio-temporal systems has been investigated thoroughly [23,29,32,33], and in many cases, equivalence has been rigorously established [24,57–59]. For a recent review, see [21].

The space–time representation of delay systems is particularly meaningful when the delay τ_D is long compared with the time scale t_c of the temporal dynamics of the system, as measured by the width of the zeroth peak in the autocorrelation [21]. In this case, there is a separation of time scales, and so it is natural to parametrize time as

$$t = n\tau_D + \sigma, \tag{3.1}$$

where *n* is an integer that counts the number of delay times since the origin, and σ is a continuous variable between 0 and τ_D that gives the position along the delay. As a result, *n* is often considered to be a discrete time and σ a continuous pseudo-spatial variable. We note that t_c is a property of the dynamics and therefore depends on β and F(x) in addition to the time scales τ_L and τ_H in equation (2.2); in practice, however, it is often the case that $t_c \approx \tau_L$ [21].

When working with delay systems, one often obtains a long time series x(t) such as the one shown in figure 2*a*. It seems that there are (and indeed one expects there to be) correlations on the order of one time delay τ_D . Plotting the time series in the space–time representation in figure 2*b* shows long-time correlations (on the order of several τ_D) as spatial structures that evolve in discrete time.

While figure 2*b* does reveal long-time correlations as spatio-temporal structures, it is clear that as *n* increases the structures are drifting to the right in σ -space. In other words, the long-time correlations occur over a time slightly larger than τ_D . This can be seen by looking at the autocorrelation of the time series, shown in figure 2*c*. The autocorrelation begins to increase near a lag of τ_D , but only reaches its peak at $\tau_D + \delta$ due to the non-instantaneous response time of the system [21]. Therefore, δ is related to the width of the zeroth autocorrelation peak t_c as well as the width of the first autocorrelation peak. Previous works have extensively studied this drift and its relation to co-moving Lyapunov exponents [25,57].

The drift is a consequence of the fact that the system is causal. The delayed term $x(t - \tau_D)$ cannot affect the dynamics before, *or even at*, the time *t*. Therefore, in figure 2*d*, we use

$$t = nT + \sigma, \tag{3.2}$$

to create space–time representations, where $T = \tau_D + \delta$ is the recurrence time and now $\sigma \in [0, T]$. When the space–time representation is done in this way, the structures are stabilized in space (i.e. they have a nearly stationary average spatial position). Indeed, it has been shown that this is often the correct moving frame in which to view the spatio-temporal behaviour of time-delayed systems [21].

4. Using the space-time representation to realize coupled oscillators in a single delay system

Recently, the space-time representation has been used to interpret a single nonlinear node with delayed feedback as a network of coupled oscillators. These experiments replace the spatial multiplexing of a traditional network (in which all nodes are updated simultaneously in parallel)



Figure 2. Illustration of the space-time representation. (*a*) Time series of the delayed system in equation (2.2). (*b*) Space-time representation of the time series shown in (*a*), where $\sigma \in [0, \tau_D]$. (*c*) Autocorrelation of the time series shown in (*a*) The distance to the first autocorrelation peak is $\tau_D + \delta$. Here $\tau_D = 4$ ms and $\delta = 250 \,\mu$ s. *Inset*. Zoom in on central autocorrelation peak. The width of this peak is t_c . (*d*) Space-time representation with drift correction ($\sigma \in [0, \tau_D + \delta]$). These figures were made from a numerical simulation of equation (2.2) with $\beta = -5$, $\tau_L = 400 \,\mu$ s, $\tau_H = 10 \,\text{ms}$, $\tau_D = 4 \,\text{ms}$ and $F(x) = \sin^2(x(t) - \pi/4)$, which describes the opto-electronic oscillator shown in figure 1*b*. (Online version in colour.)

with temporal multiplexing, in which the single nonlinear element serially updates each of the nodes, which are distributed across the delay line. There are two major benefits to this network implementation: this is the *only* way to create a network of truly identical nodes, and it allows one to implement a large network without building a large number of separate physical nodes. While originally used for a hardware implementation of RC [30,31,37,60–62], these types of delay systems have since been used to study chimera states in cyclic networks [32,33] and cluster synchronization in arbitrary networks [34,35].

Because delay systems require a continuous function to describe their initial conditions, they are considered infinite-dimensional systems. However, it was noticed early on that chaotic attractors of delay systems have finite dimension in practice [10]. In trying to explain this finite dimensionality, Le Berre *et al.* conjectured that the dimension of the attractor is equal to τ_D/t_c , where t_c is the width of the zeroth peak of the autocorrelation of the chaotic time series [11]. In other words, in practice, only τ_D/t_c values are needed to specify a point on the attractor [63]. Even more, it was suggested that a delay can be thought of as a set of τ_D/t_c roughly independent time slots, such that the *k*th time slot in one delay is correlated with only the *k*th time slot in the following delay, as confirmed by the secondary peaks in the autocorrelation function (e.g. figure 2*c*). If each of these independent time slots is considered to be a 'node', one can think of the delay system as consisting of a set of τ_D/t_c independent, discrete-time nonlinear systems. This reasoning is similar to the reasoning that led to the development of the space–time representation and is particularly useful in the same types of situations, i.e. when $\tau_D \gg t_c$.

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A **377**: 20180123

Temporal discretization arises naturally in many experimental implementations of delay systems. The opto-electronic feedback system with a pulsed laser described in §2 is one such example [55,64]. Further, many experimental delay systems implement the delay line with a synchronously clocked shift register because of the ability to easily vary the delay [30,33,34,37,41, 53,65]. In these implementations, the digital delay discretizes time into steps of size $\Delta t = \tau_D/N$, where *N* is an integer. These digital delay lines apply a constant feedback for one time step Δt , then sample the system at the end of the time step. Because of the discretization, the use of the co-moving frame $T = \tau_D + \delta$ discussed in §2 is not always necessary, and we can simply use a discretized version of original space–time representation equation (3.1).

In order to reveal the link between these systems and networks, we explicitly discretize time into time steps of length Δt , and we call each time slot a network node. If Δt is chosen to be slightly less than t_c , the nodes (which span an interval Δt) are no longer roughly independent, but are now coupled through the 'inertia' due to the non-instantaneous response time of the system to which delayed feedback is applied. This response time can be described by a filter impulse response. In this way, we have a network of coupled nodes, where the strength and topology of the coupling are determined by the form of the filter impulse response. The temporal discretization Δt is chosen depending on the application and can have an important impact on the dynamics and coupling, as we discuss at the end of this section.

In order to show explicitly how the network structure arises in these cases, we consider the discretized space-time representation

$$k = nN + i, \tag{4.1}$$

where *k* is the original discrete time, *n* is an integer that counts the number of delays that have passed, $N = \tau_D / \Delta t$ is the number of time steps in a delay, and *i* is the discrete spatial variable. In our network interpretation, *n* will be the network time and *i* will be the node index. We impose this discrete space–time representation (equation (4.1)) upon the discrete-time delayed equation (2.3):

$$x^{(i)}[n] = \beta \sum_{m=-\infty}^{nN+i} h[nN+i-m]F(x[m-N]), \qquad (4.2)$$

where $N = \tau_D / \Delta t$ is the number of nodes in the network, *n* is the network time and *i* is the node index. We can then split up this summation as follows:

$$x^{(i)}[n] = S^{(i)}[n] + C^{(i)}[n],$$
(4.3)

$$S^{(i)}[n] = \beta \sum_{m=-\infty}^{(n-1)N+i} h[nN+i-m]F(x[m-N])$$
(4.4)

$$C^{(i)}[n] = \beta \sum_{m=(n-1)N+i+1}^{nN+i} h[nN+i-m]F(x[m-N]).$$
(4.5)

Further insight into the meaning of $S^{(i)}[n]$ can be provided by a concrete example. Here we consider a single-pole low-pass filter described by [42]

$$h(t) = \tau_L^{-1} e^{-t/\tau_L} u(t), \tag{4.6}$$

where u(t) is the Heaviside step function, as depicted in figure 3*a*. This is the response that one would use, for example, when solving the Ikeda equation, equation (1.1). In this case, equation (4.4) becomes

$$S^{(i)}[n] = \beta e^{-\tau_D/\tau_L} x^{(i)}[n-1].$$
(4.7)

Equation (4.7) shows that $S^{(i)}[n]$ is a self-feedback term with a weight w_h that depends on the form of h(t). In general when the delay is long relative to the filter time scales, $w_h \rightarrow 0$, as is clear from equation (4.7) for the particular case of a low-pass filter where $w_h = e^{-\tau_D/\tau_L}$.

and



Figure 3. Impulse response for (*a*) single-pole low-pass filter and (*b*) two-pole band-pass filter. The poles are real in both cases.



Figure 4. Illustration of the coupling term in the space-time representation of delay systems (second term in equation (6.6)) (*a*) when $\tau_D \approx \tau_L$ and (*b*) when $\tau_D \gg \tau_L$ when the coupling is implemented by a band-pass filter. The colouring indicates the strength of the coupling h[k] from the shaded nodes ($x^{(p)}[n-1]$) to the node represented by the black rectangle ($x^{(i)}[n]$). Red shading represents positive coupling, blue negative coupling and white no coupling. In (*a*), the coupling spans two full time steps (n - 1 and n - 2), and so this should not be considered a network. In (*b*), however, the coupling is significant over only a small range (from $p - k_{\Delta}$ to p) and so for almost all nodes *i* the coupling comes from nodes only at time step n - 1. Therefore, this can be considered to be a network. (Online version in colour.)

In order to interpret $C^{(i)}[n]$, we perform a simple change of variables p = m - nN in equation (4.5) to obtain

$$C^{(i)}[n] = \beta \sum_{p=i+1-N}^{i} h[i-p]F(x^{(p)}[n-1]).$$
(4.8)

Therefore, $C^{(i)}[n]$ is a coupling term: the summation 'couples' the values of $x^{(p)}[n-1]$ (weighted by *h*) to the value of $x^{(i)}[n-1]$ to determine $x^{(i)}[n]$.

Equation (4.3) along with equations (4.7) and (4.8) now resembles a network equation: each node *i* is coupled to all the other nodes through the coupling weights *h*. However, this should not yet be considered a network. We recall that the superscript on *x* denotes a node index and must be in the range [0, N - 1]; however, in equation (4.8) *p* runs from i + 1 - N to *i*, which can include negative values. Physically, this means that the coupling summation runs over some *x*-values at time n - 2 in addition to those from time n - 1. This is illustrated in figure 4*a*, where the black rectangle denotes $x^{(i)}[n]$ and the shaded region denotes the *x*-values that are coupled to $x^{(i)}[n - 1]$ by $C^{(i)}[n]$ to determine $x^{(i)}[n]$.

In cases where the delay $\tau_D = N \Delta t$ is long (relative to the filter time scales), the filter impulse response is significant for only a small range, from $i - k_{\Delta}$ to i, where $k_{\Delta} \ll N$ is a small number of time steps (determined by the form of h[k]) above which $h[k_{\Delta}]$ is negligible. For long delays, we



Figure 5. Illustration of the adjacency matrices for (a) the low-pass filter and (b) band-pass filter. The adjacency matrix is cyclically symmetric due to the time invariance of the filter. (Online version in colour.)

can approximate equation (4.3) as

$$x^{(i)}[n] = w_h x^{(i)}[n-1] + \beta \sum_{p=i-k_\Delta}^i h[i-p]F(x^{(p)}[n-1]),$$
(4.9)

where the superscript denotes the node number and the number in square brackets denotes the discrete network time.

Equation (4.9) is now an exact correspondence with the standard network equation

$$x^{(i)}[n] = G(x^{(i)}[n-1]) + \sum_{j=1}^{N} A_{ij}F(x^{(j)}[n-1]),$$
(4.10)

where G(x) is a function that describes the self-feedback and A_{ij} is the weighted network adjacency matrix. By comparing equations (4.9) and (4.10), $G(x)=w_hx$. The filter impulse response h(t) is the equivalent of the adjacency matrix; it determines the strength and topology of the coupling.

For concreteness in demonstration, we now present the adjacency matrices induced by two simple but common impulse responses: the low-pass filter and the band-pass filter. The singlepole low-pass filter response is given by equation (4.6). The adjacency matrix that corresponds with this low-pass filter is given by

$$A_{ij}^{\text{LP}} = \beta \frac{\Delta t}{\tau_L} \begin{cases} e^{-(i-j)\Delta t/\tau_L} & \text{if } 0 \le i-j \le k_\Delta \\ 0 & \text{otherwise} \end{cases}.$$
 (4.11)

A depiction of this adjacency matrix is shown in figure 5*a*. We note that all couplings are positive and that the network is a directed ring. Another common type of filtering is the two-pole bandpass filter, which has impulse response [42]

$$h_{\rm BP}(t) = \frac{(1/\tau_L)e^{-t/\tau_L} - (1/\tau_H)e^{-t/\tau_H}}{1 - \tau_L/\tau_H}u(t), \tag{4.12}$$

where τ_H is the high-pass filter time constant and τ_L is again the low-pass filter time constant, depicted in figure 3*b*. This impulse response corresponds to the filtering in equation (2.2). The corresponding adjacency matrix is

$$A_{ij}^{\rm BP} = \beta \frac{\Delta t}{1 - \tau_L / \tau_H} \begin{cases} \tau_L^{-1} e^{-(i-j)\Delta t / \tau_L} - \tau_H^{-1} e^{-(i-j)\Delta t / \tau_H} & \text{if } 0 \le i-j \le k_\Delta \\ 0 & \text{otherwise.} \end{cases}$$
(4.13)

A depiction of this adjacency matrix is shown in figure 5*b*. We note that the network is again a directed ring; however, some of the couplings are now negative. Time-invariant filters, such as the two discussed above, will lead to ring networks, and the ring is directed due to



Figure 6. Graphical comparison between two RC implementations: a classical RNN architecture (left), and a delay dynamics based Reservoir. (Online version in colour.)

causality. However, networks with arbitrary topologies can be created by the introduction of a time-dependent filter, as we discuss in §7.

Here, we make a note about the design of these network experiments and the choice of Δt relative to the time scales τ_L and τ_D . If $\Delta t < \tau_L$, the (time-invariant) filter impulse response will couple the nodes in a cyclically symmetric adjacency matrix, with the coupling radius and coupling strength determined by the form of the impulse response. If $\Delta t \gg \tau_L$, no coupling will be induced by the filtering, and the system will consist of completely independent but identical nodes. Further, the quantity τ_D/τ_L should be large for the network interpretation to hold in general. If τ_D/τ_L is not large, then for a significant fraction of nodes the $C^{(i)}[n]$ includes terms from both time n - 1 and time n - 2 as shown in figure 4*a*. The fraction of nodes for which this is the case tends to zero as $\tau_D/\tau_L \to \infty$.

5. Reservoir computing with delayed feedback

RC is a recently proposed brain-inspired processing technique, corresponding to a simplified version of conventional recurrent neural network (RNN) concepts. It was independently proposed in the machine learning community under the naming Echo State Network (ESN) [66] and in the brain cognitive research community as Liquid State Machine [67]. It was later unified with the now adopted name, RC [68,69]. The generic architecture of an RC system is thus rather conventional (figure 6), consisting of:

- An input layer aimed at expanding the input information to be RC-processed onto each node of the RNN;
- An internal network having a recurrent connectivity thus potentially possessing complex internal dynamics depending on the spectral radius of its connectivity matrix;
- And an output layer intended to extract the computed result from the global observation of the network response, typically performing a linear combination of the different internal state variables of the network.

The most important difference of RC compared to conventional RNN consists in the restriction of the learning process (i.e. finding the optimal synaptic weights for the nodes and layer connectivity) to the output layer only. The input layer and the internal network connecting weights are usually set at random and remain fixed during both training and operation. This makes the learning phase of RC much more computationally efficient than in RNN (since learning is reduced to a linear regression problem in RC). In many situations, the effective computational power of RC has been found comparable, or in some cases even better than, their standard RNN counterpart.

One major technological challenge of neuromorphic computing is however to imagine and design a physical hardware implementing its specific concepts, instead of translating them into algorithms to be programmed in standard, however structurally unmatched, digital processors. The generally recognized poor energy efficiency of artificial intelligence (involving dedicated supercomputers, or energy greedy computer farms) is indeed related to the fact that brain computing concepts have to be adapted into Turing von Neumann machines, whose architecture and principles of operation are actually very far from what we have learned from the brain. Up to now, there is essentially no other easily available and dedicated computing platform capable of efficiently running artificial intelligence techniques. Turing von Neumann machines are practically the only effectively working solution today for investigating AI.

An essential problem when one wants to design a dedicated hardware implementation of neural network processing concepts is the difficulty to physically fabricate a well-controlled threedimensional dynamical network, as nature easily does with any brain. Based on the assumption that what matter are the dynamical complexity and the high phase space dimension, but not the internal structure itself of the reservoir network, the EU project PHOCUS (PHOtonic liquid state machine based on delay CoUpled Systems) started in 2010 with the objective to demonstrate the RC implementation suitability of nonlinear delay dynamics. Delay dynamics have thus been proposed as a way to replace a neural network architecture in the implementation of the RC concepts, with a first successful demonstration through an electronic delay system mimicking the Mackey–Glass dynamics [30]. To do so, extensive use of the space–time analogy of delay dynamics was made in order to properly adapt the RC processing rules previously used in networks of dynamical nodes (and effectively always programmed or simulated with digital processors).

Figure 6 shows on the left a standard network-based RC processing (ESN), whereas the right figure displays its analogue based on nonlinear delayed feedback dynamics for the reservoir. The experimental set-up first proposed for photonic RC is precisely the one depicted in figure 1*b*, in which an external signal is superimposed at the rf input port of the Mach–Zehnder.

(a) Input layer

The input information in standard RNN is expanded into the network according to spatial multiplexing: the coordinates of the original input vector $\mathbf{v}[n] \in \mathbb{R}^Q$ is expanded through the multiplication with the input connectivity matrix $W^I \in \mathbb{R}^N \times \mathbb{R}^Q$. Each node $i = 0 \cdots N - 1$ of the network is thus receiving an input signal $u^{(i)}[n]$:

$$u^{(i)}[n] = \sum_{q=1}^{Q} w_{iq}^{I} v_{q}[n].$$
(5.1)

When one is making use of a delay dynamics instead of network of nodes, time division multiplexing is naturally adopted to address the virtual nodes *i* distributed in time all along the recurrence time *T*. The required temporal waveform which will need to be injected into the delay dynamics, reads as follows:

$$u(t) = \sum_{i=0}^{N-1} \left[\sum_{q=1}^{Q} w_{iq}^{I} v_{q}[n] \right] p_{\Delta t}(t - nT - i\Delta t),$$
(5.2)

where $p_{\Delta t}(t)$ is a sample and hold function. It is a temporal window being unity from time t = 0 to $t = \Delta t$ and zero everywhere else. The duration Δt is the sampling period, or differently speaking, also the temporal spacing between two virtual nodes in the recurrence time interval *T*. The scalar signal u(t) is practically programmed in an arbitrary waveform generator, it has the shape of a piecewise constant signal for each sample $i = 0 \cdots N - 1$ of each time slot of duration Δt . When dividing u(t) into sequences of *N* samples, and stacking horizontally these vectors of length *N* for

each consecutive discrete time *n*, one obtains the space–time representation of the input signal, as depicted in figure 2*c*.

(b) Reservoir layer

A transient dynamic is then triggered in the reservoir due to the injection of the information signal $u^{(i)}[n]$ or u(t). For the ESN, this transient is ruled by the following discrete time update rule, from time (n - 1) to time n:

$$x^{(i)}[n] = F\left[\sum_{j=1}^{N} w_{ij}^{R} x^{(j)}[n-1] + \rho \cdot u^{(i)}[n]\right],$$
(5.3)

where $W^R \in \mathbb{R}^N \times \mathbb{R}^N$ is the internal connectivity matrix of the Reservoir. $F[\cdot]$ is a nonlinear function (usually a sigmoïd, e.g. a hyperbolic tangent, in classical ESN), and ρ is a scaling factor weighting the input signal defined in equation (5.1).

In the case of a delay reservoir, the update rule is similar to equation (2.1), except the delay dynamics is now non-autonomous. The input waveform defined in equation (5.2) is indeed superposed on the delayed feedback. It is thus contributing directly to a nonlinear transient in the delay dynamics phase space, with a contributing weight ρ :

$$x(t) = h(t) * F(x(t - \tau_D) + \rho \cdot u(t)) = \int_{-\infty}^{t} h(t - t') F[x(t' - \tau_D) + \rho \cdot u(t)] dt'.$$
 (5.4)

One could notice that the delay reservoir, compared to the discrete-time ESN, is continuous in time. The definition of virtual spatial nodes, and their discretization, is experimentally introduced through the sampling period Δt from equation (5.2). The adjacency matrices represented in figure 5*a*,*b* then correspond to the internal connectivity matrix W^R used for the ESN.

The time scale Δt is very important as it has to be properly tuned with respect to the internal short time τ_L of the delay dynamics. Optimal processing efficiency of the delay reservoir is empirically found for $\Delta t \simeq \tau_L/5$. This highlights a compromise between:

- Δt should not be too short, otherwise adjacent nodes have nearly identical dynamical behaviour because they are too strongly coupled through the delay dynamics inertia (the reservoir response to the input data would also be too small in amplitude, since it would be strongly filtered; this has detrimental signal-to-noise ratio impacts in the RC processing);
- the adjacent nodes could be too decoupled when Δt is too large; if they would be too far one from each other, they would allow each stepwise transition of the input information to reach an asymptotic state independently of the farther past.

(c) Output layer

The last operation in RC concerns the read-out layer, consisting of a linear combination of the reservoir internal states $x^{(i)}[n]$. This step aims to provide the expected computational result. The read-out operation generates an output vector $\mathbf{y}[n] \in \mathbb{R}^M$, which components read as follows for the ESN:

$$y_m[n] = \sum_{i=0}^{N-1} w_{mi}^O x^{(i)}[n].$$
(5.5)

The same equation holds in the case of a delay reservoir, where however the node state $x^{(i)}[n]$ corresponds to the extraction of a virtual node state in the delay reservoir, through the sampling of x(t). The signal defined by equation (5.4), is sampled to provide $x(t_k)$, with $t_k = k \cdot \Delta t$, k being defined as in equation (4.1).

The coefficients of the linear combination (i.e. the elements w_{mi}^O of the read-out matrix $W^O \in \mathbb{R}^M \times \mathbb{R}^N$) are determined by a learning task. In the case of supervised learning, one



Figure 7. Graphical illustration of the RC processing steps in the case of a speech recognition task, performed with an optoelectronic delay oscillator used as a Reservoir with 400 virtual nodes. Each input cochleagram consists of 86 frequency components, the (colour encoded) energy content of which is evolving over the duration of the spoken digit (this duration N_I amounts here to 88 steps in n). (Online version in colour.)

0

3

6

0

simply applies a ridge regression to an ill-posed problem for a set of known data couples, {(*reservoir response*₁ = A_1 , *target read* – *out*₁ = \tilde{B}_1), $l = 1 \cdots L$ }. This corresponds to a training set of L couples of temporal data (evolution of the discrete time n), each having a duration N_1 . $A_1 \in \mathbb{R}^N \times \mathbb{R}^{N_1}$ is thus the concatenation of the reservoir state vector { $x^{(i)}[n] | i = 0 \cdots N - 1$, $n = 1 \cdots N_1$ }, and $\tilde{B}_l \in \mathbb{R}^M \times \mathbb{R}^{N_l}$ is the same concatenation for the corresponding target vectors $\tilde{\mathbf{y}}[n]$. The learning requires one to consider all reservoir responses A_1 for the different elements of the training set, which are gathered into a matrix A (of dimension $N \times (\sum N_l)$). The RC outputs $W^O A$ are expected to provide the correct corresponding answers \tilde{B} (of dimension $M \times (\sum N_l)$ where, \tilde{B} is the concatenation of the target matrices \tilde{B}_l): $\tilde{B} = W^O A$. The ridge regression can be applied to solve this ill-posed problem, through the following formula giving the optimal read-out matrix: $W_{\text{opt}}^O = \tilde{B}A^T (AA^T - \lambda I)^{-1}$, where the superscript T holds for the matrix transposition operation, λ is the small regression parameter, I is the $N \times N$ identity matrix, and the matrix inversion can be calculated through a Moore–Penrose algorithm.

RC has already achieved many successes, revealing its computational potential both in ESN numerical simulations [70,71], and also in physical hardware implementation. Successful physical hardware implementations have been based on delay dynamics [30,31,60,72–74], but also more recently they have employed real spatially extended photonic systems [75,76].

Figure 7 illustrates the previously described RC processing steps, in the case the processing of a classification problem (speech recognition), as performed with an optoelectronic delay dynamical system [31]. It makes an extensive use of the space–time representation for delay dynamical systems.

6. The continuum limit

sparse input connectivity

 $(N \times Q)$ -matrix, W^I

100

200

300

400

20 40 60

expected target read-out

used for the training

 $(M \times N_l)$ - matrix \widetilde{B}_l

cochleagram of a spoken digit:

input vector $v_a(n)$ (time-frequency

speech decomposition, $Q \times N_l$

80

60

read-out test result $\mathbf{y}(n)$

displayed in a $(M \times N_i)$ -

concatenated matrix B_1

20

40

60

86

3

6

20 40

86

Networks can also be realized using the space-time representation in the case of fully analogue delay lines, such as those that rely on the finite propagation speed of light. Such a system can also be well approximated by the discrete-time systems discussed in §3 by taking the limit

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A 377: 20180123

that $\Delta t/\tau_D \rightarrow 0$ [32,33]. In these situations, time is continuous, so we return to the space–time representation given by equation (3.2). This allows us to think of a continuum of nodes that are labelled by their position σ and evolve in discrete time *n*.

The realization of a network follows very much along the lines of §3, but in continuous time rather than discrete. Therefore, the summations will be replaced by integrals, and we will have to account for the drift δ in the space–time representation. What follows is an elaboration of the presentation contained in [33].

We begin by analysing equation (2.1) from the perspective of the space–time representation by setting $t = nT + \sigma$, where *n* is an integer that counts the number of drift-corrected delays $T = \tau_D + \delta$ that have passed since the origin, and $\sigma \in [0, T]$ is the node's position in pseudo-space. Re-writing equation (2.1) with this change of variables results in

$$x_n(\sigma) = \beta \int_{-\infty}^{nT+\sigma} h(nT+\sigma-t') F\left(x(t'-\tau_D)\right) dt'.$$
(6.1)

We can then separate the integral into two domains as follows:

$$x_n(\sigma) = S_n(\sigma) + C_n(\sigma), \tag{6.2}$$

$$S_n(\sigma) = \beta \int_{-\infty}^{(n-1)T+\sigma} h(nT+\sigma-t') F\left(x(t'-\tau_D)\right) dt'$$
(6.3)

$$C_n(\sigma) = \beta \int_{(n-1)T+\sigma}^{nT+\sigma} h(nT+\sigma-t') F\left(x(t'-\tau_D)\right) dt'.$$
(6.4)

Further insight into the meaning of $S_n(\sigma)$ can be provided by a concrete example, so that we can evaluate the integral. Here we consider the simplest filter, a single-pole low-pass filter described by $h(t) = \tau_l^{-1} e^{-t/\tau_L} u(t)$ (equation (4.6)). In this case, equation (6.3) becomes

$$S_n(\sigma) = \beta e^{-T/\tau_L} x_{n-1}(\sigma). \tag{6.5}$$

The meaning of $S_n(\sigma)$ is now clear: it is a self-feedback term (from the state *x* at the spatial position σ at discrete time n - 1 to the state at the spatial position σ at discrete time n) with a strength determined by the form of h(t).

In order to interpret $C_n(\sigma)$, we make a change of variables $t'' = t' + \delta - nT$:

$$C_n(\sigma) = \beta \int_{\sigma-\tau_D}^{\sigma+\delta} h(\sigma+\delta-t'') F\left(x_{n-1}(t'')\right) dt''.$$
(6.6)

Therefore, $C_n(\sigma)$ is a coupling term: the integral 'couples' the values of $x_{n-1}(t'')$ to the value of $x_{n-1}(\sigma)$ to determine $x_n(\sigma)$.

When the delay τ_D is long (relative to the filter time scale), the filter impulse response is significant for only a small range, from $\sigma - \Delta$ to $\sigma + \delta$, where $\Delta \ll \tau_D$ is a short time (determined by the form of h(t)) above which h(t) is negligible. For long delays, we can approximate equation (6.6) as

$$C_n(\sigma) \approx \beta \int_{\sigma-\Delta}^{\sigma+\delta} h(\sigma+\delta-t'') F\left(x_{n-1}(t'')\right) dt''.$$
(6.7)

Equations (6.6) and (6.7) reveal the network structure that results from viewing the system with long delay through the space–time representation. The system can be interpreted as a continuum of discrete-time nodes whose position (node index) is given by σ . Each node is coupled to its neighbours within a distance Δ on the left and δ on the right through the system's impulse response h(t), as shown in figure 4. Importantly, the coupling term in equation (6.7) includes only nodes from time step n - 1 for almost all nodes σ since $\Delta \ll \tau_D$. Indeed, in the limit $\tau_L/\tau_D \rightarrow 0$, the fraction of nodes whose input coupling spans two time steps vanishes. It is clear from equation (6.7) that h(t) determines both the coupling strength and the coupling width. The particular form of h(t) plays a crucial role in the types of dynamics that the system can exhibit.

and

7. Chimeras in systems with delayed feedback

Chimeras and RC surprisingly share a temporal and a spatial coincidence. They were 'temporally' discovered and invented, respectively, in the early 2000s [38,66,67], and they were 'geographically' connected to delay dynamics during the Delayed Complex Systems conference DCS'12, a decade later. Since delay dynamics had demonstrated the ability to emulate a virtual network of neurons in RC applications, a straightforward challenge was to also confirm the relevance of this network emulation for the experimental observation of chimera patterns. Moreover, chimeras had just been experimentally discovered in 2012, in set-ups modelled by spatio-temporal equations [6,7]. A chimera state in a delayed dynamical system was first observed in 2013 [54], illustrating that networks of dynamical nodes can indeed be emulated by a delayed dynamical system.

A chimera is an unexpected solution arising in a homogeneous network of identically coupled oscillators. A chimera is a symmetry breaking solution: Even though the network is structurally homogeneous (identical oscillators and coupling topology) the network is split into two groups. In one group, the oscillators behave coherently, while the oscillators in the other group behave incoherently. One of the models used to numerically explore chimera solutions is the network of continuously distributed coupled Kuramoto oscillators, defined as follows:

$$\frac{\partial \phi}{\partial t} = \omega_0 + \int G(x - \xi) \cdot \sin[\alpha + \phi(t, x) - \phi(t, x - \xi)] \,\mathrm{d}\xi. \tag{7.1}$$

This governs the dynamics of the phases $\phi(t, x)$ of the oscillators that are continuously distributed in space, ω_0 being their natural angular frequency. Oscillators have coupled phases according to a sine nonlinear dependency of the coupling (with an important coupling offset α), depending on the relative phase difference between the two coupled oscillators at position x and $x - \xi$. Each phase coupling is weighted by a distance-dependent factor $G(x - \xi)$, which is typically vanishing beyond a certain coupling distance (sometimes referred as to the coupling radius) defined by the shape of $G(\cdot)$. The phase dynamics is thus ruled by the contribution of the coupling with all the other oscillators, as the integral term in equation (7.1) covers the entire space of the network. Chimera solutions of such an equation typically consist of coherent clusters, in which oscillators in the same cluster are synchronized with the same phase, and incoherent clusters, in which oscillators are completely desynchronized with chaotically fluctuating phases.

It is then interesting to compare qualitatively the integral term in equation (7.1), with the one derived in equation (6.7). As previously discussed and as it can be also inferred from the comparison with the network of Kuramoto oscillators, one can clearly identify the specific role of h(t), when it is considered in the space–time representation of the delay dynamical variable $x_n(\sigma)$ as derived in equations (6.2)–(6.6). The impulse response h(t) clearly controls the coupling strength and the coupling distance within the virtual network of dynamical nodes. The nonlinear function F(x) plays the role of the nonlinear coupling between the amplitudes of the virtual nodes.

Figure 8 reports typical chimera patterns obtained experimentally with nonlinear delay dynamics. It shows both the temporal waveform during growth and stabilization of the pattern, as well as the space–time representation in the (σ , n)-plane, with colour encoding of the waveform amplitude. The space–time picture clearly shows the sustained chimera pattern along the horizontal virtual space domain. It consists of a flat plateau (blue colour) surrounded by a chaotic sea (red and orange colours), with which it coexists, filling in a balanced and stable way the shared spatial domain. The figure also shows two possible solutions (single-headed and two-headed chimera), obtained with the same experimental parameters, but produced from different noisy initial conditions. Depending on the temporal parameters (hence the properties of the coupling function h(t) as depicted in figure 3b, e.g. the actual values of τ_L and τ_H relatively to τ_D), one can obtain a highly multistable dynamics of chimera patterns [33].

To comment more into the details under which conditions chimera solutions can be obtained in delay dynamics, it is worth mentioning that indeed h(t) requires a bandpass profile. There are many different arguments to explain this requirement. The first is related to the carrier



Figure 8. Experimental record of single- and two-headed chimera solutions generated in delay dynamics. The two central graphs show the space-time representation of the chimera solution, as they grow and are then stabilized. The side graphs, left and right, are temporal waveforms showing parts of the chimera solution, during the initial transient (birth of the chimera from the noisy background; lower time-traces, covering a few hundreds of time recurrences in the delayed feedback loop), and during the stabilized part at the end of the full record (upper time-traces; covering approximately two recurrent times $T = \tau_D + \delta$). (Online version in colour.)

waveform of a chimera pattern over the virtual spatial domain $[n(\tau_D + \delta); (n + 1)(\tau_D + \delta)]$, which is necessarily a stable period-1 (delay) carrier waveform, and not a period-2 carrier waveform as usually concerned in the period-doubling bifurcation cascade typically known for delay dynamics. To allow for such a stable period-1 carrier waveform, the band-pass character for h(t)is necessary (stable period-1 pattern have been analysed e.g. in [77]), since the low-pass one is known to lead to unstable period-1 pattern, as was reported in [25] about the 'coarsening' of any forced initial pattern in the virtual spatial domain. Last but not least, one could also mention that with a fixed τ_L , the impulse response with τ_H (band-pass) necessarily exhibits a broader width than without the presence of τ_H (low-pass). This remark is in line with the known fact that chimera states are favoured when the coupling range is extended (i.e. beyond the classical case of nearest-neighbour coupling only, which does not allow for chimera states).

There are also specific requirements on the nonlinear coupling function F(x) for obtaining chimera solutions. This is illustrated in figure 9, where both the nonlinear function profile is represented, and next to it, with the same vertical scaling, the temporal chimera waveform. From the standard fixed point analysis for a nonlinear map defined by the same function F(x), one can notice the following:

- The nonlinear function operates around an average value centred along a positive slope of F(x), between two extrema, where an unstable fixed point for the map is located (middle black circle);
- The high amplitude chaotic part of the chimera waveform corresponds to the sharp maximum of F(x), and it develops a chaotic motion along this maximum, essentially on the negative slope side and centred around an unstable fixed point (upper-right black circle);
- The low amplitude plateau of the chimera waveform corresponds to a stable fixed point (lower-left black disc) of the map, along a weak negative slope, thanks to the presence of a broad minimum.

This remark points out that F(x) must have an asymmetric shape that results in a sharp maximum and a broad minimum. This was experimentally obtained in [33] with the Airy function provided by a low finesse Fabry–Pérot resonator, which is providing a nonlinear transformation of the wavelength of a dynamically tunable laser diode, into the output optical intensity of the Fabry–Pérot.

The space–time representation was recently found not to be restricted to a single virtual space dimension. Indeed, adding a second delay much larger than the first one, and acting in parallel to it, enabled two-dimensional chimera to be obtained in delay systems. Among various solutions



Figure 9. Features of the nonlinear coupling function for obtaining chimera patterns in delay dynamics. Left: nonlinear function profile F(x), with a dotted first bisector line highlighting the fixed points for a map $x_{n+1} = F(x_n)$. Right: amplitude correspondence in the temporal chimera waveform x(t). (Online version in colour.)

observed in this two-delay system, one could observe chaotic islands surrounded by a calm sea, or its contrary, a flat plateau island in the middle of a chaotic sea [78].

8. Arbitrary networks of coupled maps

Section 3 described the realization of circularly symmetric networks in a single nonlinear system with delayed feedback. In these experiments, the network nodes were time slots of length Δt , where $\Delta t \ll \tau_D$, and the coupling between nodes was due to the inherent bandwidth of the electronics. This inherent bandwidth was described using a time-invariant infinite impulse response filter; the time invariance results in a circularly symmetric network. However, equation (4.2) does not require the impulse response to be time-invariant. In this section, we describe recent work that uses a digital filter with a time-varying impulse response to realize arbitrary networks in an experimental delay system [34].

There are two modifications of previous systems necessary in order to obtain a network with arbitrary topology: (*a*) the inherent circularly symmetric coupling due to the (time-invariant) bandwidth limitations of the system must be removed, and (*b*) the desired coupling must be implemented by an appropriately designed filter with a time-dependent impulse.

(a) Removing the inherent circularly symmetric coupling

There are two convenient options for removing the inherent circularly symmetric coupling due to the time-invariant bandwidth limitations of the system.

(I) Perhaps the most straightforward way to remove the coupling due to the bandwidth limitations of the system is to extend the Δt described in §5. This can be done in the pulsed laser system described by equation (2.3) by choosing the pulse repetition rate $f_r = N/\tau_D \ll 1/\tau_L$. In this case, the filter response decays before the next pulse arrives, and so the system reduces to the *N*-dimensional map:

$$x[k] = \beta F(x[k-N]), \qquad (8.1)$$

where k is the discrete time. This map equation requires the specification of N different initial conditions, but the trajectory of each initial condition is completely independent of the trajectories of the others. Therefore, (8.1) can be thought of as a set of N completely independent but *truly identical* oscillators using the space–time representation:

$$x^{(i)}[n] = \beta F(x^{(i)}[n-1]), \tag{8.2}$$

where $i = k \mod N$ is the oscillator number and n is the network time.

(II) An easier-to-implement experiment that displays the same map dynamics is obtained by using a CW laser and sample-and-hold electronics that are clocked at a rate f_r . Synchronously clocked shift registers have long been used to implement delays in experimental set-ups because of the ease of varying the delay [30,33,34,37,41,43,53,65]. Such a system can also be described by equation (2.3). However, in previous experiments, the clock rates have typically been chosen so that the discrete-time nature of the digital delay line minimally impacts the dynamics; that is, the sampling time $\Delta t = 1/f_r$ has typically been much smaller than any other dynamical time scale,

and so the digital delay line is a good approximation of an analogue delay. In these cases, the experiment is well described by equation (2.1). Here, we intentionally choose a sampling time that is much longer than the other dynamical time scales in the system, but still shorter than the time delay $\tau_D = N\Delta t$. With this choice of clock rate, the dynamics of the system is well described by equation (8.2).

(b) Implementing the desired adjacency matrix

The systems described in the last few paragraphs create N identical, uncoupled nodes using a single delayed dynamical system. In order to couple the nodes together in a network, we must implement a filter that can be described by a time-varying impulse response. This is easiest to do with a digital filter, since in this case we are not restricted by what can be easily implemented by analogue components.

It is convenient to implement both the delay and the digital filter on a single device such as a field-programmable gate array (FPGA). In this case, the filter can be *acausal* in the sense that we can implement the following:

$$x[k] = \sum_{m=-\infty}^{(k+N-i-1)} h[k-m;k]F(x[m-N]),$$
(8.3)

where the impulse response *h* is explicitly written as a function of the discrete time *k* to denote that it is varying in time. The acausality of the filter is necessary in order to permit couplings to node *i* from nodes j > i.

The impulse response of the digital filter necessary to implement a given network is determined by the adjacency matrix A_{ij} that describes the network as follows:

$$h[m;k] = \begin{cases} \beta & \text{if } m = k \\ \sigma A_{ij} & \text{if } m \neq k \text{ and } m = k - i + j \\ 0 & \text{otherwise,} \end{cases}$$
(8.4)

where $i = k \mod N$ and j is an integer between 0 and N - 1.

When the digital filter described by the impulse response in equation (8.4) is implemented and equation (8.3) is written in the space–time representation, we obtain

$$x^{(i)}[n] = \beta F(x^{(i)}[n-1]) + \sigma \sum_{j} A_{ij} F(x^{(j)}[n-1]),$$
(8.5)

which describes a network of discrete-time oscillators that are coupled by the arbitrary adjacency matrix A_{ij} .

There are two adjustments, then, that need to be made to the systems described in §4 in order to realize an arbitrary network of coupled oscillators in a single delay system:

- (a) Time must be discretized in such a way as to break the nearest-neighbour coupling that would otherwise be induced by the bandwidth limitations of the system.
- (b) A filter with a time-dependent impulse response must be used in order to obtain a network topology that is not cyclically symmetric. This filter must also be acausal to allow for the construction of all possible networks (e.g. to couple node N 1 to node 0).

(c) Experimental examples

This technique has been used to implement arbitrary networks in an optoelectronic feedback loop [34]. An illustration of our experiment is shown in figure 10*a*. Light of constant intensity is emitted from a fiber-coupled CW laser. The light passes through an electro-optic intensity modulator, which serves as a nonlinearity. The light is converted to an electrical signal by a photodiode and sampled at a frequency f_r by the FPGA via an analogue to digital converter (ADC). The FPGA



Figure 10. Experimental schematic for realizing arbitrary networks using a single nonlinearity with temporal multiplexing (*a*) through a single delay and time-dependent filtering (*b*) through multiple time delays that are switched on and off in time. Both illustrations are different ways of viewing the same experiment. (Online version in colour.)

implements the delay and the time-dependent digital filtering, and outputs the feedback electrical signal through a digital to analog converter (DAC). This signal is amplified and fed back to the modulator, completing the feedback loop.

One example of a network that can be implemented using this experimental technique is shown in figure 11*a*. Clearly, the network is not rotationally symmetric, so it cannot be implemented by a time-invariant filter. Figure 11*b* shows experimental time series measured from the system depicted in figure 10. If we reorganize this time series according to the space-time interpretation given by equation (4.1), we obtain figure 11*c*, which clearly shows cluster synchronization: nodes 0, 1, 8 and 9 form one synchronized cluster, and nodes 2–7 form the other synchronized cluster.

This network is particularly interesting because it displays an unexpected type of cluster synchronization [35]. It had previously been shown that nodes that could be permuted among each other by a symmetry operation could form synchronous clusters [8]. Later, it was shown that in some cases, symmetry clusters could be combined to form non-symmetric synchronous clusters. This was shown first in Laplacian networks [79] then later in more general networks [80]. Figure 11*a* is a simple example of such a network, as nodes 3 and 6 cannot be permuted with



Figure 11. Experimental observation of cluster synchronization using a single feedback loop implementation of an arbitrary network from [35]. (*a*) Illustration of the network that was implemented. The shading indicates the cluster synchronous state that was observed: nodes that are the same colour are in the same synchronous cluster. All nodes are truly identical. (*b*) Experimentally measured time series. The dotted black lines indicate one network time step. (*c*) Space–time representation of the time series shown in (*b*). The cluster synchronous network dynamics are periodic with period two. The parameters used for the measurement are $\beta = 1.10$, $\sigma = 0.16$, $\phi = \pi/4$. (Online version in colour.)



Figure 12. Experimental observation of a chimera state using a single feedback loop implementation of a globally coupled network. (*a*) Illustration of the globally coupled network that was implemented in [34]. The shading indicates the pattern of synchrony that was observed: nodes that are the same colour are in the same synchronous cluster. All nodes are truly identical. (*b*) Experimentally measured time series. The dotted black lines indicate one network time step. (*c*) Space–time representation of the time series shown in (*b*). The parameters used for the measurement are $\beta = 2.3$, $\sigma = 0.23$, $\phi = \pi/4$. (Online version in colour.)

nodes 2, 4, 5 or 7; yet, the red cluster still synchronizes, as shown in figure 11*c*. These experiments confirm the stability of such so-called equitable partition cluster synchronization [35].

As mentioned in §7, a chimera state is a dynamical state of a network in which the nodes split up into a coherent set and an incoherent set despite the fact that they are all identical and coupled identically [38,39]. The chimeras in §7 were observed in a network with circularly symmetric coupling and many nodes. Using the system shown in figure 10, we were able to observe a chimera state in a 5 node globally coupled network [34]. The experimental results are shown in figure 12. The globally coupled network and associated adjacency matrix are shown in figure 12*a*. The colours denote the set of synchronized nodes: the blue nodes (0,2,3) are in the coherent set, and the red and black nodes are desynchronized both with the blue nodes and with each other. All nodes are truly identical. Figure 12*b* shows the time series, where the dotted lines denote the increments of the network time step *n*. Figure 12*c* shows the space–time representation of the time series, which clearly shows that nodes 0,2 and 3 are synchronized, and nodes 1 and 4 are desynchronized from all nodes. Linear stability calculations confirm that these chimera states are linearly stable [34]. There is an alternative (but equivalent) way to view the technique used to create arbitrary networks that does not involve acausal filtering. This perspective is described in detail in [34]. Here, the acausal filter is replaced by multiple delays that are switched on and off as a function of time in order to implement the desired network. The idea of using multiple time delays to create a more interesting network was pioneered for the purpose of RC [37]; however, in this case, each delay was always switched on, resulting again in a circularly symmetric network (albeit with longer range connections than with a single delay). Switching the additional delays on and off in time breaks the time-invariance (and therefore circular symmetry of the network) and allows an arbitrary network topology. The time-dependent switching is determined according to the following recipe:

- 1. The time delay of length *N* is always switched ON. This is the feedback time delay and is multiplied by β . This delay is modelled by the first term in equation (8.5).
- 2. Time delays of length N + i j are switched ON if $A_{ij} = 1$, where $i = k \mod N$ is the active node. These time delays determine the coupling and are summed then multiplied by σ . This is modelled by the second term in equation (8.5).
- 3. All other time delays are switched OFF.

Digital time delays and switches are easily implemented in FPGA, making this a particularly powerful implementation because the networks are easy to reconfigure. A schematic of such an experiment is shown in figure 10*b*.

9. Conclusion and outlook

The realization of networks of coupled oscillators is a challenging experimental task because of the difficulty and expense of obtaining, coupling and measuring a large number of identical oscillators. In this paper, we have reviewed recently developed techniques that overcome these obstacles by implementing the network in a single nonlinear delay system through temporal multiplexing. These techniques offer the additional benefit, impossible in other network implementations, that the oscillators are *truly identical* since they are all implemented in the same physical hardware. These delay networks were first developed for their vast potential as a physical implementation of RC with low cost and high speed. In addition to these important information processing applications, delay networks are also opening up entirely new avenues of research in basic experimental science, as exemplified by the observation of novel one- and two-dimensional chimera states and cluster synchronization. These techniques, first conceived only in 2011, are still in their infancy and continue to stimulate basic and applied research.

Future work might explore the use of experimental arbitrary networks for hardware-based reservoir computing, where a time-dependent filter impulse response might allow for the use of a shorter time delay and therefore for faster information processing. This technique can also be used for the experimental study of a variety of fundamental questions of network dynamics, including the impact of targeted perturbations on network dynamics [81,82], the effect of heterogeneities on network dynamics [83,84], the control of network dynamics [85] and the impact of noise on network dynamics.

While the delay systems themselves are often continuous-time systems, the space-time representation causes delay networks to be discrete in time. Research is currently under way to allow the realization of continuous-time networks in a single delay by adopting the multiple time delay implementation of arbitrary networks, shown in figure 10*b*. Importantly, this technique is not reliant on opto-electronics: one could replace the optics with any system of interest. This might be useful for building prototypes for large networks of coupled oscillators when the oscillators are expensive, such as in the case of power grids. It may also allow for the experimental study of large networks of truly identical oscillators in situations where the oscillators are rarely identical in practice (e.g. biological systems such as neurons). This permits the study of the impact of heterogeneity on the network dynamics.

Data accessibility. This article has no additional data.

Competing interests. We declare we have no competing interests.

Funding. No funding has been received for this article.

Acknowledgements. J.D.H. and R.R. were supported by the U.S. Office of Naval Research ONR Grant No. N000141612481. L.L. thanks the support from the ANR project BiPhoProc (ANR-14-OHRI-0002-02), and the EIPHI program (ANR-17-EURE-0002).

References

- Bassett DS, Sporns O. 2017 Network neuroscience. Nat. Neurosci. 20, 353. (doi:10.1038/ nn.4502)
- Davidson E, Levin M. 2005 Gene regulatory networks. Proc. Natl Acad. Sci. USA 102, 4935– 4935. (doi:10.1073/pnas.0502024102)
- 3. Motter AE, Myers SA, Anghel M, Nishikawa T. 2013 Spontaneous synchrony in power-grid networks. *Nat. Phys.* **9**, 191. (doi:10.1038/nphys2535)
- 4. Jaeger H. 2002 Tutorial on training recurrent neural networks, covering BPPT, RTRL, EKF and the 'echo state network' approach, vol. 5. GMD-Forschungszentrum Informationstechnik Bonn.
- Boccaletti S, Latora V, Moreno Y, Chavez M, Hwang D-U. 2006 Complex networks: structure and dynamics. *Phys. Rep.* 424, 175–308. (doi:10.1016/j.physrep.2005.10.009)
- 6. Hagerstrom AM, Murphy TE, Roy R, Hövel P, Omelchenko I, Schöll E. 2012 Experimental observation of chimeras in coupled-map lattices. *Nat. Phys.* **8**, 658. (doi:10.1038/nphys2372)
- Tinsley MR, Nkomo S, Showalter K. 2012 Chimera and phase-cluster states in populations of coupled chemical oscillators. *Nat. Phys.* 8, 662. (doi:10.1038/nphys2371)
- Pecora LM, Sorrentino F, Hagerstrom AM, Murphy TE, Roy R. 2014 Cluster synchronization and isolated desynchronization in complex networks with symmetries. *Nat. Commun.* 5, 4079. (doi:10.1038/ncomms5079)
- Lang R, Kobayashi K. 1980 External optical feedback effects on semiconductor injection laser properties. *IEEE J. Quantum Electron.* 16, 347–355. (doi:10.1109/JQE.1980.1070479)
- Farmer JD. 1982 Chaotic attractors of an infinite-dimensional dynamical system. *Physica D* 4, 366–393. (doi:10.1016/0167-2789(82)90042-2)
- Le Berre M, Ressayre E, Tallet A, Gibbs HM, Kaplan DL, Rose MH. 1987 Conjecture on the dimensions of chaotic attractors of delayed-feedback dynamical systems. *Phys. Rev. A* 35, 4020. (doi:10.1103/PhysRevA.35.4020)
- Franz AL, Roy R, Shaw LB, Schwartz IB. 2008 Effect of multiple time delays on intensity fluctuation dynamics in fiber ring lasers. *Phys. Rev. E* 78, 016208. (doi:10.1103/Phys RevE.78.016208)
- 13. Erneux T. 2009 *Applied delay differential equations*, vol. 3. Berlin, Germany: Springer Science & Business Media.
- 14. Erneux T, Javaloyes J, Wolfrum M, Yanchuk S. 2017 Introduction to focus issue: time-delay dynamics. *Chaos* 27, 114201. (doi:10.1063/1.5011354)
- Mackey MC, Glass L. 1977 Oscillation and chaos in physiological control systems. *Science* 197, 287–289. (doi:10.1126/science.267326)
- 16. Kuang Y. 1993 *Delay differential equations: with applications in population dynamics,* vol. 191. New York, NY: Academic Press.
- 17. Ikeda K. 1979 Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system. *Opt. Commun.* **30**, 257–261. (doi:10.1016/0030-4018(79)90090-7)
- Chow S-N, Hale JK, Huang W. 1992 From sine waves to square waves in delay equations. Proc. R. Soc. Edinb. Sect. A 120, 223–229. (doi:10.1017/S0308210500032108)
- Weicker L, Erneux T, D'Huys O, Danckaert J, Jacquot M, Chembo Y, Larger L. 2012 Strongly asymmetric square waves in a time-delayed system. *Phys. Rev. E* 86, 055201. (doi:10.1103/Phys RevE.86.055201)
- 20. Müller D, Otto A, Radons G. 2018 Laminar chaos. *Phys. Rev. Lett.* **120**, 084102. (doi:10.1103/PhysRevLett.120.084102)
- 21. Yanchuk S, Giacomelli G. 2017 Spatio-temporal phenomena in complex systems with time delays. *J. Phys. A* **50**, 103001. (doi:10.1088/1751-8121/50/10/103001)
- 22. Arecchi FT, Giacomelli G, Lapucci A, Meucci R. 1992 Two-dimensional representation of a delayed dynamical system. *Phys. Rev. A* **45**, R4225. (doi:10.1103/PhysRevA.45.R4225)

- 23. Giacomelli G, Meucci R, Politi A, Arecchi FT. 1994 Defects and spacelike properties of delayed dynamical systems. *Phys. Rev. Lett.* **73**, 1099. (doi:10.1103/PhysRevLett.73.1099)
- 24. Yanchuk S, Giacomelli G. 2014 Pattern formation in systems with multiple delayed feedbacks. *Phys. Rev. Lett.* **112**, 174103. (doi:10.1103/PhysRevLett.112.174103)
- 25. Giacomelli G, Marino F, Zaks MA, Yanchuk S. 2012 Coarsening in a bistable system with long-delayed feedback. *Europhys. Lett.* **99**, 58005. (doi:10.1209/0295-5075/99/58005)
- Javaloyes J, Ackemann T, Hurtado A. 2015 Arrest of domain coarsening via antiperiodic regimes in delay systems. *Phys. Rev. Lett.* 115, 203901. (doi:10.1103/PhysRevLett.115. 203901)
- 27. Giacomelli G, Marino F, Zaks MA, Yanchuk S. 2013 Nucleation in bistable dynamical systems with long delay. *Phys. Rev. E* 88, 062920. (doi:10.1103/PhysRevE.88.062920)
- 28. Marino F, Giacomelli G. 2017 Pseudo-spatial coherence resonance in an excitable laser with long delayed feedback. *Chaos* 27, 114302. (doi:10.1063/1.5006744)
- Faggian M, Ginelli F, Marino F, Giacomelli G. 2018 Evidence of a critical phase transition in purely temporal dynamics with long-delayed feedback. *Phys. Rev. Lett.* **120**, 173901. (doi:10.1103/PhysRevLett.120.173901)
- 30. Appeltant L *et al.* 2011 Information processing using a single dynamical node as complex system. *Nat. Commun.* **2**, 468. (doi:10.1038/ncomms1476)
- Larger L, Soriano MC, Brunner D, Appeltant L, Gutiérrez JM, Pesquera L, Mirasso CR, Fischer I. 2012 Photonic information processing beyond turing: an optoelectronic implementation of reservoir computing. *Opt. Express* 20, 3241–3249. (doi:10.1364/OE.20.003241)
- Larger L, Penkovsky B, Maistrenko Y. 2013 Virtual chimera states for delayed-feedback systems. *Phys. Rev. Lett.* 111, 054103. (doi:10.1103/PhysRevLett.111.054103)
- Larger L, Penkovsky B, Maistrenko Y. 2015 Laser chimeras as a paradigm for multistable patterns in complex systems. *Nat. Commun.* 6, 7752. (doi:10.1038/ncomms8752)
- Hart JD, Schmadel DC, Murphy TE, Roy R. 2017 Experiments with arbitrary networks in time-multiplexed delay systems. *Chaos* 27, 121103. (doi:10.1063/1.5016047)
- 35. Siddique AB, Pecora L, Hart JD, Sorrentino F. 2018 Symmetry-and input-cluster synchronization in networks. *Phys. Rev. E* **97**, 042217. (doi:10.1103/PhysRevE.97.042217)
- Hart JD, Zhang Y, Roy R, Motter AE. 2019 Topological control of synchronization patterns: trading symmetry for stability. *Phys. Rev. Lett.* 122, 058301. (doi:10.1103/PhysRev Lett.122.058301)
- Martinenghi R, Rybalko S, Jacquot M, Chembo YK, Larger L. 2012 Photonic nonlinear transient computing with multiple-delay wavelength dynamics. *Phys. Rev. Lett.* 108, 244101. (doi:10.1103/PhysRevLett.108.244101)
- 38. Kuramoto Y, Battogtokh D. 2002 Coexistence of coherence and incoherence in nonlocally coupled phase oscillators. *Nonlin. Phenom. Complex Syst.* **5**, 380–385.
- Abrams DM, Strogatz SH. 2004 Chimera states for coupled oscillators. *Phys. Rev. Lett.* 93, 174102. (doi:10.1103/PhysRevLett.93.174102)
- Panaggio MJ, Abrams DM, Ashwin P, Laing CR. 2016 Chimera states in networks of phase oscillators: the case of two small populations. *Phys. Rev. E* 93, 012218. (doi:10.1103/Phys RevE.93.012218)
- 41. Hart JD, Bansal K, Murphy TE, Roy R. 2016 Experimental observation of chimera and cluster states in a minimal globally coupled network. *Chaos* **26**, 094801. (doi:10.1063/1.4953662)
- 42. Oppenheim AV, Willsky AS, Hamid-Nawab S. 1996 *Signals & systems*, vol. 18, pp. 19–21.Upper Saddle River, NJ: Prentice-Hall, Inc.
- 43. Gibbs HM, Hopf FA, Kaplan DL, Shoemaker RL. 1981 Observation of chaos in optical bistability. *Phys. Rev. Lett.* **46**, 474. (doi:10.1103/PhysRevLett.46.474)
- 44. Neyer A, Voges E. 1982 Dynamics of electrooptic bistable devices with delayed feedback. *IEEE J. Quantum Electron.* **18**, 2009–2015. (doi:10.1109/JQE.1982.1071487)
- Peil M, Jacquot M, Chembo YK, Larger L, Erneux T. 2009 Routes to chaos and multiple time scale dynamics in broadband bandpass nonlinear delay electro-optic oscillators. *Phys. Rev. E* 79, 026208. (doi:10.1103/PhysRevE.79.026208)
- Kouomou YC, Colet P, Larger L, Gastaud N. 2005 Chaotic breathers in delayed electro-optical systems. *Phys. Rev. Lett.* 95, 203903. (doi:10.1103/PhysRevLett.95.203903)
- 47. Callan KE, Illing L, Gao Z, Gauthier DJ, Schöll E. 2010 Broadband chaos generated by an optoelectronic oscillator. *Phys. Rev. Lett.* **104**, 113901. (doi:10.1103/PhysRevLett.104.113901)

- Ravoori B, Cohen AB, Sun J, Motter AE, Murphy TE, Roy R. 2011 Robustness of optimal synchronization in real networks. *Phys. Rev. Lett.* 107, 034102. (doi:10.1103/Phys RevLett.107.034102)
- Hagerstrom AM, Murphy TE, Roy R. 2015 Harvesting entropy and quantifying the transition from noise to chaos in a photon-counting feedback loop. *Proc. Natl Acad. Sci. USA* 112, 9258– 9263. (doi:10.1073/pnas.1506600112)
- 50. Yao XS, Maleki L. 1996 Optoelectronic microwave oscillator. *JOSA B* **13**, 1725–1735. (doi:10.1364/JOSAB.13.001725)
- Goedgebuer J-P, Levy P, Larger L, Chen C-C, Rhodes WT. 2002 Optical communication with synchronized hyperchaos generated electrooptically. *IEEE J. Quantum Electron.* 38, 1178–1183. (doi:10.1109/JQE.2002.802025)
- 52. Argyris A *et al.* 2005 Chaos-based communications at high bit rates using commercial fibreoptic links. *Nature* **438**, 343–346. (doi:10.1038/nature04275)
- 53. Murphy TE *et al.* 2010 Complex dynamics and synchronization of delayed-feedback nonlinear oscillators. *Phil. Trans. R. Soc. A* **368**, 343–366. (doi:10.1098/rsta.2009.0225)
- 54. Larger L. 2013 Complexity in electro-optic delay dynamics: modelling, design and applications. *Phil. Trans. R. Soc. A* **371**, 20120464. (doi:10.1098/rsta.2012.0464)
- Larger L, Lacourt P-A, Poinsot S, Hanna M. 2005 From flow to map in an experimental high-dimensional electro-optic nonlinear delay oscillator. *Phys. Rev. Lett.* 95, 043903. (doi:10.1103/PhysRevLett.95.043903)
- 56. Ikeda K, Matsumoto K. 1989 Information theoretical characterization of turbulence. *Phys. Rev. Lett.* **62**, 2265. (doi:10.1103/PhysRevLett.62.2265)
- 57. Giacomelli G, Politi A. 1996 Relationship between delayed and spatially extended dynamical systems. *Phys. Rev. Lett.* **76**, 2686. (doi:10.1103/PhysRevLett.76.2686)
- Wolfrum M, Yanchuk S. 2006 Eckhaus instability in systems with large delay. *Phys. Rev. Lett.* 96, 220201. (doi:10.1103/PhysRevLett.96.220201)
- 59. Yanchuk S, Giacomelli G. 2015 Dynamical systems with multiple long-delayed feedbacks: multiscale analysis and spatiotemporal equivalence. *Phys. Rev. E* **92**, 042903. (doi:10.1103/PhysRevE.92.042903)
- 60. Paquot Y, Duport F, Smerieri A, Dambre J, Schrauwen B, Haelterman M, Massar S. 2012 Optoelectronic reservoir computing. *Sci. Rep.* **2**, 287. (doi:10.1038/srep00287)
- 61. Brunner D, Soriano MC, Mirasso CR, Fischer I. 2013 Parallel photonic information processing at gigabyte per second data rates using transient states. *Nat. Commun.* **4**, 1364. (doi:10.1038/ncomms2368)
- Antonik P, Haelterman M, Massar S. 2017 Brain-inspired photonic signal processor for generating periodic patterns and emulating chaotic systems. *Phys. Rev. Appl.* 7, 054014. (doi:10.1103/PhysRevApplied.7.054014)
- 63. Lepri S, Giacomelli G, Politi A, Arecchi FT. 1994 High-dimensional chaos in delayed dynamical systems. *Physica D* **70**, 235–249. (doi:10.1016/0167-2789(94)90016-7)
- 64. Grapinet M, Udaltsov V, Jacquot M, Lacourt P-A, Dudley JM, Larger L. 2008 Experimental chaotic map generated by picosecond laser pulse-seeded electro-optic nonlinear delay dynamics. *Chaos* **18**, 013110. (doi:10.1063/1.2839910)
- 65. Williams CRS, Sorrentino F, Murphy TE, Roy R. 2013 Synchronization states and multistability in a ring of periodic oscillators: experimentally variable coupling delays. *Chaos* **23**, 043117. (doi:10.1063/1.4829626)
- 66. Jaeger H. 2001 The 'echo state'Âİ approach to analysing and training recurrent neural networks-with an erratum note. *Bonn, Germany: German National Research Center for Information Technology GMD Technical Report* **148**, 13.
- Maass W, Natschläger T, Markram H. 2002 Real-time computing without stable states: a new framework for neural computation based on perturbations. *Neural Comput.* 14, 2531–2560. (doi:10.1162/089976602760407955)
- 68. Verstraeten D, Schrauwen B, Stroobandt D, Van Campenhout J. 2005 Isolated word recognition with the liquid state machine: a case study. *Inf. Process. Lett.* **95**, 521–528. (doi:10.1016/j.ipl.2005.05.019)
- Lukoševičius M, Jaeger H. 2009 Reservoir computing approaches to recurrent neural network training. *Comput. Sci. Rev.* 3, 127–149. (doi:10.1016/j.cosrev.2009.03.005)
- 70. Jaeger H, Haas H. 2004 Harnessing nonlinearity: predicting chaotic systems and saving energy in wireless communication. *Science* **304**, 78–80. (doi:10.1126/science.1091277)

- Pathak J, Hunt B, Girvan M, Lu Z, Ott E. 2018 Model-free prediction of large spatiotemporally chaotic systems from data: a reservoir computing approach. *Phys. Rev. Lett.* **120**, 024102. (doi:10.1103/PhysRevLett.120.024102)
- 72. Brunner D, Soriano MC, Mirasso CR, Fischer I. 2013 Parallel photonic information processing at gigabyte per second data rates using transient states. *Nat. Commun.* **4**, 1364. (doi:10.1038/ncomms2368)
- 73. Haynes ND, Soriano MC, Rosin DP, Fischer I, Gauthier DJ. 2015 Reservoir computing with a single time-delay autonomous boolean node. *Phys. Rev. E* **91**, 020801. (doi:10.1103/Phys RevE.91.020801)
- Larger L, Baylón-Fuentes A, Martinenghi R, Udaltsov VS, Chembo YK, Jacquot M. 2017 Highspeed photonic reservoir computing using a time-delay-based architecture: million words per second classification. *Phys. Rev. X* 7, 011015. (doi:10.1103/physrevx.7.011015)
- 75. Brunner D, Fischer I. 2015 Reconfigurable semiconductor laser networks based on diffractive coupling. *Opt. Lett.* **40**, 3854–3857. (doi:10.1364/OL.40.003854)
- 76. Bueno J, Maktoobi S, Froehly L, Fischer I, Jacquot M, Larger L, Brunner D. 2018 Reinforcement learning in a large-scale photonic recurrent neural network. *Optica* 5, 756–760. (doi:10.1364/OPTICA.5.000756)
- Weicker L, Erneux T, D'Huys O, Danckaert J, Jacquot M, Chembo Y, Larger L. 2013 Slowfast dynamics of a time-delayed electro-optic oscillator. *Phil. Trans. R. Soc. A* 371, 20120459. (doi:10.1098/rsta.2012.0459)
- Brunner D, Penkovsky B, Levchenko R, Schöll E, Larger L, Maistrenko Y. 2018 Twodimensional spatio-temporal complexity in dual-delayed nonlinear feedback systems: chimeras and dissipative solitons. *Chaos* 28, 103106. (doi:10.1063/1.5043391)
- 79. Sorrentino F, Pecora LM, Hagerstrom AM, Murphy TE, Roy R. 2016 Complete characterization of the stability of cluster synchronization in complex dynamical networks. *Sci. Adv.* **2**, e1501737. (doi:10.1126/sciadv.1501737)
- Schaub MT, O'Clery N, Billeh YN, Delvenne J-C, Lambiotte R, Barahona M. 2016 Graph partitions and cluster synchronization in networks of oscillators. *Chaos* 26, 094821. (doi:10.1063/1.4961065)
- Menck PJ, Heitzig J, Marwan N, Kurths J. 2013 How basin stability complements the linearstability paradigm. *Nat. Phys.* 9, 89–92. (doi:10.1038/nphys2516)
- 82. Menck PJ, Heitzig J, Kurths J, Schellnhuber HJ. 2014 How dead ends undermine power grid stability. *Nat. Commun.* **5**, 3969. (doi:10.1038/ncomms4969)
- 83. Sorrentino F, Pecora L. 2016 Approximate cluster synchronization in networks with symmetries and parameter mismatches. *Chaos* **26**, 094823. (doi:10.1063/1.4961967)
- Nishikawa T, Motter AE. 2016 Symmetric states requiring system asymmetry. *Phys. Rev. Lett.* 117, 114101. (doi:10.1103/PhysRevLett.117.114101)
- 85. Sorrentino F, Di Bernardo M, Garofalo F, Chen G. 2007 Controllability of complex networks via pinning. *Phys. Rev. E* **75**, 046103. (doi:10.1103/PhysRevE.75.046103)